## Introduction

In our HL mathematics syllabus, we learn about two methods that can be used to solve systems of linear equations: the substitution method and the Gaussian elimination method. A system of linear equations is a set of linear equations that you solve at the same time[1]. These systems of linear equations can have multiple variables and are invaluable in their applications, from aiding in business decisions to analytical chemistry, to even personal decisions such as planning a diet.

However, both the substitution method and the Gaussian elimination method have their limitations. I noticed that these methods could not solve a system of linear equations when too many variables and equations are involved, as it would take too long of a time to do so. This caused a necessity to employ other methods to solve such systems of linear equations. After a consultation with my mathematics teacher about this problem, she told me that there are many other methods to solving these systems of linear equations, other than by substitution and Gaussian elimination, which are able to solve these systems of linear equations using matrices. I then took it upon myself to research on these methods as I was curious about the many methods that my teacher told me about.

A system of linear equations can always be represented using a matrix. This is useful as it can greatly simplify the solving of systems of linear equations. When you solve a matrix which represents a system of linear equations, you can easily solve the system of linear equations by substitution. When the system of linear equations has too many variables and too many equations, the matrix which represents them becomes very large. Solving a large matrix is very tedious, complicated and would take a long time to solve in order to obtain the solution to the system of linear equations. Thus, solving of large matrices are critical in their uses to solve such complicated systems of linear equations.

When searching online, I noticed that there were two categories of methods that could be used in order to solve such large matrices: direct methods and iterative methods. Thus, I asked myself about the differences between these two types of methods, in their efficiencies to solve these large matrices.

Direct methods are methods which provide an answer which gives the exact answer straight away, while iterative methods are methods that give an approximate answer, and the more times the iterative method is repeated, the closer the approximate answer obtained is to the true value[9].

I also noticed when researching online that LU decomposition is a typical example of a direct method that can be used in order to solve large matrices, while the Gauss-Seidel method is a typical example of an iterative method that can be used to solve large matrices. Hence I decided to use LU decomposition and the Gauss-Seidel method as representations of direct and iterative methods for the solving of matrices and thus, by extension, in their abilities to solve systems of linear equations with too many variables and equations.

**Overview**

In this investigation, I will do the following:

1. Show how systems of linear equations can be represented by matrices
2. A brief summary of the substitution method and the Gaussian elimination method
3. An explanation of LU Decomposition and how it can be used to solve large matrices
4. An explanation of the Gauss-Seidel method and how it can be used to solve large matrices
5. A table of comparison showing the advantages and disadvantages of direct and iterative methods, based on LU decomposition and the Gauss-Seidel method
6. Conclusion

**Systems of linear equations and matrices**

Assuming that a system has enough equations to solve for the unknowns, we can say that the unknowns in the system can always be solved using a square matrix, with the same number of equations as unknowns. In any of such cases, the system of linear equations can be represented by an $n \times n$ matrix[4]:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & \ddots & & & a_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$$

or

$$Ax = b$$

For illustration purposes, we will be using a $3 \times 3$ matrix for this paper:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Where $x_1, x_2, x_3$ are the unknown variables, and the values of $b_n$ (where n is a positive integer) are the answers to the system of linear equations represented by their corresponding row.

**Gaussian elimination and algebraic substitution**

When matrices are small, Gaussian Elimination and algebraic substitution can be used to solve the system of linear equations.

Algebraic substitution, is the most straightforward way of solving systems of linear equation. In Secondary School, this was the method taught to us for the solving of systems of linear equations. However, it can be extremely tedious, especially when the number of variables goes beyond 2 or 3.
Example (with only 2 variables):

$$x + 4y = 63 \ (1)$$
$$x + y = 18 \ \ (2)$$

From (1): Let $x = 63 - 4y$

Substituting this into (2), we have:

$$63 - 4y + y = 18$$
$$63 - 3y = 18$$
$$3y = 45$$
$$y = 15$$

When $y = 15$,

$$x + 15 = 18$$
$$\therefore x = 3, y = 15$$

Gaussian elimination uses elementary row operations and an augmented matrix, in order to solve for the system of linear equations. Elementary row operations are a group of procedures that can be done with the rows of a matrix[8]. These operations are: row-switching, multiplication and addition[8], as learned in out HL Mathematic syllabus. The goal of Gaussian elimination is to make the matrix sparser and hence, easier to solve. Using elementary row operations, we can create a triangle of zeroes in the bottom left hand corner of the augmented matrix, thus making it sparser. By making a matrix sparser, we are able to reduce the number of unnecessary variables of the system of linear equations represented by the matrix.
Example:

$$A = \begin{pmatrix} -2 & 1 & 3 & 9 \\ 1 & 2 & 3 & 14 \\ 1 & 1 & -2 & -3 \end{pmatrix}$$

Using elementary row operations, matrix $A$ can be reduced to:

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & \frac{9}{5} & \frac{37}{5} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$